

sequences $(a_1, a_2, a_3, \dots, a_n, \dots)$ and series $(\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots)$

n-th partial sum $(s_n = a_1 + a_2 + a_3 + \dots + a_n) - \sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n$

$\sum_{n=1}^{\infty} a_n$ is convergent, $a_n \geq 0$, iff $s_n \leq M$

geometric series, base is a fraction

$a + ar^1 + ar^2 + \dots + ar^n + \dots$, convergent if $|r| < 1$, $a \neq 0$, converges to $\frac{\text{first term}}{1 - \text{ratio}}$, $\frac{a}{1 - r}$

$$s_n = \frac{a(1 - r^n)}{(1 - r)}, \sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} s_n = \frac{a(1 - r^n)}{(1 - r)} = \frac{a}{1 - r}$$

p-series, power is less than -1

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \text{ converges if, } p > 1$$

telescoping series

$$a_n = (b_n - b_{n+1}) = \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \dots = 1 - \frac{1}{n+1}$$

converges to $\frac{1}{n}$ or b_n , $s_n = 1 - \frac{1}{n+1}$, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \rightarrow \infty} s_n \left(1 - \frac{1}{n+1}\right)$

integral test

compare $\sum_{n=1}^{\infty} a_n$ to $\int_0^{\infty} a_n dn$, must be positive, decreasing

direct comparison test ($\sum_{n=1}^{\infty} a_n$ is non-negative), use $1/(x+1) \leq 1/x$ method, consider a geo or p

$\sum_{n=1}^{\infty} a_n$ converges if $a_n \leq b_n$ and if b_n converges

$\sum_{n=1}^{\infty} a_n$ diverges if $a_n \geq b_n$ and if b_n diverges and is non-negative

limit comparison ($a_n > 0$ and $b_n > 0$ for all n being a positive integer), consider a geo or p

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = C$ and $0 < C < \infty$, then either both converge or both diverge

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$, and b_n converges, then a_n converges

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$, and b_n diverges, then a_n diverges

ratio test, use if factorial or exponent, plug in $(n + 1)$ do not just add 1 to get a_{n+1}

$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = C$, if, $C < 1$, then, a_n converges, > 1 diverges, $= 1$ inconclusive

n-th root test, use if factorial or exponent

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = C, \text{ if, } C < 1, \text{ then, } a_n \text{ converges, } > 1 \text{ diverges, } = 1 \text{ inconclusive}$$

* alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n, \text{ converges when, } u_n \geq 0, u_1 \geq u_2 \geq u_3, u_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

absolute convergence

$$\text{If } \sum |a_n| \text{ converges, } \sum a_n \text{ is absolutely convergent}$$

conditionally convergent

$$\text{If } \sum |a_n| \text{ diverges, } \sum a_n \text{ is conditionally convergent if *}$$

term-by-term differentiation and integration

you can diff. and int. the series or the summation [like (a/1-r)] to find what you want

rearrangement of terms

$$\text{A.C. - if } \sum b_n \text{ is an rearrangement of } \sum a_n, \sum a_n = \sum b_n$$

$$\text{C.C. - for any number N, you can rearrange } \sum a_n \text{ so } \sum b_n = N$$

power series, R = 1 for a geometric series

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n + \dots = \sum_{n=0}^{\infty} c_n(x-a)^n, \text{ centered at } x = a$$

$$\sum_{n=0}^{\infty} c_n(x-a)^n, 0 \leq R \leq \infty, |x-a| < R, \text{ implies absolute convergence, } = r, \text{ maybe}$$

r is radius of convergence. $|x-a| < R = (a-R, a+R)$ is the interval of convergence

theorem

$$\sum_{n=0}^{\infty} c_n(x-a)^n, \text{ suppose } \lim_{n \rightarrow \infty} \sqrt[n]{c_n} = \rho \text{ or } \lim_{n \rightarrow \infty} c_{n+1}/c_n = \rho, \text{ then } R = 1/\rho$$

taylor series, quadratic (to power 2), and cubic approximations (to power 3), not linearization

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \dots$$

remainder of taylor polynomial

$$f(x) = P_n(x) + R_n(x), \quad |R_n(x)| \leq \frac{|f^{(n+1)}(c)|_{\max} (x-a)^{n+1}}{(n+1)!}, \text{ c is between x and a}$$

multiplying

$$\text{If } A(x) = \sum_{n=0}^{\infty} a_n(x)^n \text{ and } B(x) = \sum_{n=0}^{\infty} b_n(x)^n \text{ for } |x| < R, \text{ then}$$

$$\sum_{n=0}^{\infty} c_n(x)^n = \left(\sum_{n=0}^{\infty} a_n(x)^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n(x)^n \right) \text{ converges absolutely to } A(x) \cdot B(x) \text{ for } |x| < R$$

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_{n-k} = \sum_{k=0}^n a_k b_{n-k}$$

taylor's formula, linearization, $n = n$, at $(0,0)$ $h = x$ and $k = y$ and its (cx,cy)

$$f(a+h, b+k) = f(a,b) + \underbrace{\dots + \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f|_{(a,b)}}_{\text{linearization with } f \text{ and its partial derivatives evaluated at } (a,b)} + \underbrace{\frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f|_{(a+ch, b+ck)}}_{\text{error}}$$

solving differential equations using power series

$$y(x) = a_0 + a_1x + \dots + a_nx^n + \dots$$

use this equation, the initial condition to generate a new form of the diff. equation

then find a numeric pattern to the a_n term to plug into a_nx^n , and to find y

$$\frac{x^n}{n!} = e^x \quad \frac{x^{n+1}}{n!} = \frac{x^n x}{n!} = xe^x \quad \frac{2^{n-1}x^n}{n!} = \frac{2^n x^n}{2n!} = \frac{e^{2x}}{2} \quad \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

limits/sequences/series

simplify. take the limit of a_n to see if it goes to 0

when tempted to say, behaves like, divide by n^p

combine comparison and integral tests by comparing it to something you can integrate

eliminate constants when using comparison test

indeterminate forms of limits: $0 \cdot \infty, \infty - \infty, 0/0, \infty/\infty, 0^\infty, \infty^0, 1^\infty$

when the limit is $\infty - \infty$, try to rationalize the expression. Get to $0/0, \infty/\infty$ form

when the limit is $0 \cdot \infty$, convert $(f)(g)$ into $f/(1/g)$

when the limit is $0^\infty, \infty^0, 1^\infty$, use $y = f^g \Rightarrow \ln y = g \ln f$ or $f^g = e^{g \ln f}$

remember:

$$\lim_{n \rightarrow 0} \frac{\sin x}{x} = 1, \text{ or } \lim_{n \rightarrow 0} \frac{x}{\sin x} = 1$$

$$\lim_{n \rightarrow 0} \frac{1 - \cos x}{x} = 0, \text{ or } \lim_{n \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1 \quad \lim_{n \rightarrow \infty} \sqrt[n]{x} = 1$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n} \right)^n = \lim_{n \rightarrow \infty} \left(\frac{n+x}{n} \right)^n = e^x$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{P(x)} = 1, \text{ i.e. } \sum P(x)r^n < \infty$$

$$\lim_{n \rightarrow \infty} \frac{e^x - 1}{x} = 1 \quad n! \square (n/e)^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, (-1,1)$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, (-1,1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, (-\infty, \infty)$$

$$\lim_{n \rightarrow 0} (1+x)^{1/x} = e$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, (-\infty, \infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}, [-1,1]$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, (-\infty, \infty)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, (-1,1]$$