

unit, direction = $\frac{\mathbf{v}}{|\mathbf{v}|}$ length, speed = $|\mathbf{v}|$ $\mathbf{u} = \text{proj}_{\mathbf{v}}\mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u})$

vector projection scalar projection angle between 2 lines or \mathbf{n} 's of 2 planes

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \qquad |\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \qquad \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

projectile motion $\mathbf{r}(t) = (x_0 + v_0 t \cos \theta_i)\mathbf{i} + (y_0 + v_0 t \sin \theta_i - 1/2gt^2)\mathbf{j}$

max height = $\frac{(v_0 \sin \theta)^2}{2g}$ flight time = $\frac{2v_0 \sin \theta}{g}$ range = $\frac{v_0^2 \sin 2\theta}{g}$

line, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $P_0(x_0, y_0, z_0)$

$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ $(x, y, z) = (x_0, y_0, z_0) + t(v_1, v_2, v_3)$ parametric

$\mathbf{n} \cdot P_0(x_0, y_0) = \mathbf{i}_n(x - x_0) + \mathbf{j}_n(y - y_0) = 0$, $\mathbf{n} = \nabla f \times \nabla g$ explicit

plane $\mathbf{n} \cdot P_0(x_0, y_0, z_0) = \mathbf{i}_n(x - x_0) + \mathbf{j}_n(y - y_0) + \mathbf{k}_n(z - z_0) = 0$

distance from point to a line, distance from point to plane

$$d = \frac{|\overline{PS} \times \mathbf{v}|}{|\mathbf{v}|} \qquad d = \left| \overline{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \qquad \overline{PS} \text{ is a point on the line/plane to the point in } \langle \rangle \text{ form}$$

TNB-plane for $\mathbf{r}(t)$ **T** – normal plane **N** – rectilinear plane **B** – oscillating plane

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T} \times \mathbf{N} \qquad \kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| \qquad \tau = -\frac{d\mathbf{B}}{dt} \frac{1}{|\mathbf{v}|} \cdot \mathbf{N}$$

$$\mathbf{v} = \frac{dr}{dt} \qquad \mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} \qquad \kappa_{\text{circle}} = \frac{1}{r}$$

chain rule

$$w = w(x, y, \dots) \qquad x = x(t) \qquad y = y(t) \qquad \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \dots$$

$$w = w(x, y, \dots) \qquad x = x(r, s, \dots) \qquad y = y(r, s, \dots) \qquad \frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \dots$$

implicit differentiation

$F(x, y)$ and $F(x, y) = 0$, defines y as $f(x)$, then $\frac{dy}{dx} = -\frac{F_x}{F_y}$

the gradient of f , ∇f (del 'f', nabla), ∇f gives you the direction of the highest rate of change

$$\nabla f(x, y, \dots) = \left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \dots \right)$$

the rate of change of f in any direction $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ at point P_0

$$\nabla f_{P_0} \cdot \frac{\mathbf{u}}{|\mathbf{u}|}$$

direction of most rapid change, when \mathbf{u} is in the direction of ∇f , at ∇f

direction of no change, when $\mathbf{u} \perp \nabla f$, $\langle \mathbf{a}\mathbf{i}, \mathbf{b}\mathbf{j} \rangle \perp \langle -\mathbf{b}\mathbf{i}, \mathbf{a}\mathbf{j} \rangle$ and $\langle \mathbf{a}\mathbf{i}, \mathbf{b}\mathbf{j} \rangle \perp \langle \mathbf{b}\mathbf{i}, -\mathbf{a}\mathbf{j} \rangle$

the change of f through space in (ds) units in any direction $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$

$$df = (\nabla f \cdot \mathbf{u}) \cdot ds$$

total differential

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy, \text{ for small changes in } x \text{ and } y$$

parameterization

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$$

tangent plane at a point $f(x, y, z)(P_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

tangent plane at a point $f(x, y)(P_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) - (z - z_0) = 0$$

normal line at point P

$$x = x_0 + f_x(P_0)t \quad y = y_0 + f_y(P_0)t \quad z = z_0 + f_z(P_0)t$$

linearization approximation, $n = 1$

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

magnitude of the error of the linearization

$$|E(x, y)| \leq (1/2)M(|x - x_0| + |y - y_0|)^2 \quad |f_{xx}|, |f_{yy}|, |f_{xy}| \leq M$$

$$(1/2)M(|x - x_0| + |y - y_0|)^2 \text{ is short for } (1/2!)(f_{xx}|x - x_0| + f_{xy}|x - x_0||y - y_0| + f_{yy}|y - y_0|)^2$$

extreme values

first derivative test, critical points

$$f_x = f_y = 0 \text{ or } f_x = f_y = d.n.e.$$

second derivative test

if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at critical points then you have local minimum

if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at critical points then you have local maximum

if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at critical points then you have a saddle point

if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at critical point then the test is inconclusive

boundary points

make a boundary area for the given conditions

test the end points of the sides of the boundary areas

set the derivative of the boundary sides to zero to see if they go to zero

compare all the values of all the mins and maxs at the end

lagrange multiplier

when given a multivariate function, and a constraint for the boundary, find the minimum or maximum by solving the system of equations generated by

$$\nabla f = \lambda \nabla g \text{ and } g(x, y, z) = 0$$

if solving for the minimum distance from the origin to the function, then you are solving

for the maximum distance you can expand the sphere $f(x, y, z) = x^2 + y^2 + z^2$

before it hits the constraint of $g(x, y, z) = 0$ (because its fixed) where $f = g$

it is easy to set the equation equal to lambda and then cancel the lambdas right away

if there are two constraints, $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$ then $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

partial derivatives with constrained variables

if $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$, then solve for the dependent variable, in terms of the independent variables, if not easy to solve for, do so implicitly

$$\left(\frac{\partial w}{\partial x}\right)_y = \frac{\partial(x^2 + y^2 + (x^2 + y^2)^2)}{\partial x} \quad (w, z \text{ dep.})(\text{the bottom ones } x, y \text{ are indep.})$$

Jacobian transformation, used to derive cylindrical coordinates, spherical coordinates, etc.

$$J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \quad \text{for a transformation from } f(x, y) \rightarrow f(u, v)$$

$$\iint f(x, y, z) dx dy = \iint f(u, v) |J(u, v)| du dv$$

arc length

$$\int_a^b |\mathbf{v}| dt = \int_{t_0}^t \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2} d\tau = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau, \text{ the } \mathbf{i}, \mathbf{j}, \text{ and } \mathbf{k} \text{ go away}$$

line integral

$$\int_a^b f(x, y, z) ds = \int_a^b \mathbf{r}(t) |\mathbf{v}(t)| dt$$

area in cartesian coordinates

$$\int_{y=a}^{y=b} \int_{x=x(y)} f(x, y) dx dy$$

surface area of $f(x, y, z) = c$, later filling in z for what z equals

$$\iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA \quad \mathbf{p} \text{ is a unit vector normal to the region, sometimes } \mathbf{p} = \mathbf{k}$$

surface integral of $f(x, y, z) = c$, later filling in z for what z equals, $\mathbf{p} \neq 0$

$$\iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R g(x, y, z) \sqrt{1 + f_x^2 + f_y^2} dA = \iint g(x, y, z) d\sigma$$

surface area, parameterization form

$$\iint_R |\mathbf{r}_u \times \mathbf{r}_v| du dv = \iint_S d\sigma$$

surface integral, parameterization form

$$\iint_R \mathbf{r} \langle x(u, v), y(u, v), z(u, v) \rangle |\mathbf{r}_u \times \mathbf{r}_v| du dv = \iint_S \mathbf{r}(x, y, z) d\sigma$$

volume in cartesian coordinates

$$\int_{x=a}^{x=b} \int_{y=y(x)}^{y=y(x)} \int_{z=z(x,y)}^{z=z(x,y)} f(x, y, z) dz dy dx$$

area in polar coordinates

$$\int_{\theta=a}^{\theta=b} \int_{r=r(\theta)} f(r, \theta) r dr d\theta$$

volume in cylindrical coordinates

$$\int_{\theta=a}^{\theta=b} \int_{r=r(\theta)} \int_{z=z(r,\theta)} f(r, \theta, z) r dz dr d\theta$$

volume in spherical coordinates

$$\int_{\theta=a}^{\theta=b} \int_{\phi=\phi(\theta)}^{\phi=\phi(\theta)} \int_{\rho=\rho(\theta,\phi)}^{\rho=\rho(\theta,\phi)} \rho^2 \sin \phi d\rho d\phi d\theta$$

spherical coordinates, for $f(\rho, \phi, \theta)$, $0 \leq \theta \leq 2\pi$, $0 \leq \rho \leq \pi/2$

$$\begin{aligned} x &= \rho \sin \phi \cos \theta & r &= \rho \sin \phi & \rho^2 &= x^2 + y^2 + z^2 = r^2 + z^2 \\ y &= \rho \sin \phi \sin \theta & z &= \rho \cos \phi & dV &= \rho^2 \sin \phi d\rho d\phi d\theta \end{aligned}$$

average value

$$\frac{1}{A} \iint f(x, y) dx dy \quad \frac{1}{V} \iiint f(x, y, z) dz dy dx$$

mass, use symmetry

$$m = \iint \delta dA \quad m = \iiint \delta dV$$

first moments, use symmetry

$$m_x = \iint y \delta dA \quad m_{yz} = \iiint x \delta dV$$

center of mass

$$\bar{x} = m_y / m \quad \bar{y} = m_x / m$$

moment of inertia, second moments, single integral and surface integral are the same

$$I_x = \iint y^2 \delta dA \quad I_x = \iiint (y^2 + z^2) \delta dV \quad I_0 = I_x + I_y$$

radii of gyration, same center of mass only a circle

$$R_x = \sqrt{I_x / m} \quad R_L = \sqrt{I_L / m}$$

work

$$\int \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt = \int \mathbf{F} \cdot \mathbf{T} ds = \int \mathbf{F} \cdot d\mathbf{r}$$

flow, or circulation if it is a closed loop, along $\mathbf{r}(t)$ of $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$

$$\int \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int \mathbf{F} \cdot \mathbf{T} ds$$

greene's theorem, flux across a simple closed curve $\mathbf{r}(t)$, through a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$

use the given curve to figure the limits of the integral

$$\oint M dy - N dx = \iint \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy = \iint \nabla \cdot \mathbf{F} dA = \oint \mathbf{F} \cdot \mathbf{T} ds, \text{ normal form}$$

$$\oint M dx + N dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint \nabla \times \mathbf{F} \cdot \mathbf{k} dA = \oint \mathbf{F} \cdot \mathbf{T} ds, \text{ tangential form}$$

stoke's theorem, for a simply connected open region in space with boundary

$$\iint \mathbf{F} \cdot (\pm \mathbf{r}_u \times \mathbf{r}_v) du dv = \iint \nabla \times \mathbf{F} \cdot \frac{\pm \nabla f}{|\nabla f \cdot \mathbf{p}|} dx dy = \oint \mathbf{F} \cdot d\mathbf{r} = \iint \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$$

you will need three equations, the field, the surface to integrate, and another equation that you use to determine the boundaries of the second equation

$\mathbf{F} \cdot d\mathbf{r}$

$$\mathbf{r} = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}, \quad d\mathbf{r} = -r \sin \theta d\theta \mathbf{i} + r \cos \theta d\theta \mathbf{j}$$

$$\mathbf{F} \cdot d\mathbf{r} = \langle r \sin \theta \mathbf{i} - r \cos \theta \mathbf{j} \rangle \cdot \langle -r \sin \theta d\theta \mathbf{i} + r \cos \theta d\theta \mathbf{j} \rangle = -2r^2 d\theta$$

divergence theorem, net outward flux in the direction of the normal

$$\iiint \nabla \cdot \mathbf{F} dV = \int \mathbf{F} \cdot \mathbf{n} d\sigma \quad \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}, \text{ divergence}$$

flux of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ across curve $f(x, y)$

$$\int \mathbf{F} \cdot \mathbf{n} ds = \int \left(M \frac{dy}{ds} - N \frac{dx}{ds} \right) ds = \text{counter} \int M dy - N dx \quad \mathbf{n} = \frac{\nabla f}{|\nabla f|}$$

flux of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ across surface

$$\iint \mathbf{F} \cdot \frac{\pm \nabla f}{|\nabla f \cdot \mathbf{p}|} dA \quad \mathbf{p} \text{ is the normal to the plane of the region, sometimes } \mathbf{k}$$

flux of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ across surface, parameterization form

$$\iint \mathbf{F} \cdot (\pm \mathbf{r}_u \times \mathbf{r}_v) du dv$$

flux of a three dimensional vector field

$$\iint \mathbf{F} \cdot \mathbf{n} d\sigma = \iint \mathbf{F} \cdot \pm \frac{\nabla f}{|\nabla f \cdot \mathbf{p}|} dA \quad d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA, \text{ the surface area differential}$$

if $\nabla \times \mathbf{F} = 0$, then counter $\int \mathbf{F} \cdot d\mathbf{r} = 0$ for a simply connected open region in space

flux density, divergence of a vector at a point

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

closed path

$$\oint \mathbf{F} d\mathbf{r} = 0 \text{ and the field is conservative}$$

to prove that the differential form is exact, or the $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative

solve the system of partial differential equations generated by

$$f_x = M \quad f_y = N \quad f_z = P, \text{ to find } f$$

integrate f_x to get $f(x, y, z)$ with a constant $g(y, z)$, then differentiate that to get f_y

and set that equal to your original f_y to get $g(y, z)$, then integrate that to get $f(x, y, z)$

with a constant $h(z)$, then differentiate that and set it equal to your f_z to get $h(z)$

k component of circulation density or of curl

$$\text{curl } \mathbf{F} \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

a field is conservative if the curl of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = 0 \quad \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

explicit $y = f(x)$

implicit $F(x, y) = 0$

parametric $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \quad x = x(t) \quad y = y(t)$

to find the level curve at a point, set the function equal to the values plugged into f

kepler's first law (conic section)

$$e = \frac{r_0 v_0^2}{GM} - 1 \qquad r = \frac{(1+e)r_0}{1+e \cos \theta} \text{ (polar)}$$

$$e = 0, \text{ circle} \mid 0 < e < 1, \text{ ellipse } e = \sqrt{1 - \frac{b^2}{a^2}} \mid e = 1, \text{ parabola} \mid e > 1, \text{ hyperbola } e = \sqrt{1 + \frac{b^2}{a^2}}$$

kepler's second law (equal area)

$$\frac{dA}{dt} = \frac{1}{2} r_0 v_0 = \frac{1}{2} r^2 \dot{\theta}$$

kepler's third law (time-distance)

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$